AQA
Realising potential

2021 ASSESSMENT MATERIALS

A-level
MATHS
Integration (Topic H)

Total number of marks: 41

1 Given that

$$\int_0^{10} \mathrm{f}(x)\,\mathrm{d}x = 7$$

deduce the value of

$$\int_0^{10} \left(\mathrm{f}(x) + 1\right) \mathrm{d}x$$

Circle your answer.

[1 mark]

-3 $\qquad$ 7 $\qquad$ 8 $\qquad$ 17

6 Four students, Tom, Josh, Floella and Georgia are attempting to complete the indefinite integral

$$\int \frac{1}{x}\,\mathrm{d}x \qquad \text{for } x > 0$$

Each of the students' solutions is shown below:

Tom	$\int \frac{1}{x}\,\mathrm{d}x = \ln x$
Josh	$\int \frac{1}{x}\,\mathrm{d}x = k\ln x$
Floella	$\int \frac{1}{x}\,\mathrm{d}x = \ln Ax$
Georgia	$\int \frac{1}{x}\,\mathrm{d}x = \ln x + c$

6 (a) (i) Explain what is wrong with Tom's answer.

[1 mark]

It doesn't include the constant of integration which is needed as there is a family of solutons to the integral

6 (a) (ii) Explain what is wrong with Josh's answer.

[1 mark]

Josh put in a multiplicative constant when it is not need

6 (b) Explain why Floella and Georgia's answers are equivalent.

[2 marks]

$\ln Ax = \ln A + \ln x$ and $\ln A$ is just a constant so their two answers are equivalent

16 (a) $y = e^{-x}(\sin x + \cos x)$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$

Simplify your answer.

[3 marks]

$$\frac{dy}{dx} = e^{-x}(\cos x - \sin x) - e^{-x}(\sin x + \cos x)$$

$$= e^{-x}(\cos x - \cos x - 2\sin x) = -2e^{-x}\sin x$$

16 (b) Hence, show that

$$\int e^{-x} \sin x \, dx = a e^{-x}(\sin x + \cos x) + c$$

where a is a rational number.

[2 marks]

$$\frac{d}{dx} e^{-x}(\sin x + \cos x) = -2e^{-x} \sin x$$

So $\int e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x}(\sin x + \cos x) + c$

5 Use integration by substitution to show that

$$\int_{-\frac{1}{4}}^{6} x\sqrt{4x+1} \, dx = \frac{875}{12}$$

Fully justify your answer.

[6 marks]

$u = 4x + 1 \Rightarrow x = \frac{u-1}{4}$

$\frac{du}{dx} = 4$

So $\int_{-\frac{1}{4}}^{6} x\sqrt{4x+1} \, dx = \int_{0}^{25} \frac{u-1}{4} \sqrt{u} \left(\frac{1}{4} du\right) = \int_{0}^{25} \frac{1}{16}\left(u^{3/2} - u^{1/2}\right) du$

$= \frac{1}{16}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_0^{25} = \frac{1}{16}\left(\frac{2}{5}(5)^5 - \frac{2}{3}(5)^3\right) = \frac{1}{16}\left(1250 - \frac{250}{3}\right)$

$= \frac{1}{16}\left(\frac{3500}{3}\right) = \frac{875}{12}$

5 Solve the differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\ln x}{x^2 t} \qquad \text{for } x > 0$$

given $x = 1$ when $t = 2$

Write your answer in the form $t^2 = \mathrm{f}(x)$

[7 marks]

$$\int t \, dt = \int \frac{\ln x}{x^2} \, dx \qquad u = \ln x \quad \frac{dv}{dx} = x^{-2} \quad \frac{du}{dx} = \frac{1}{x} \quad v = -x^{-1}$$

$$\frac{t^2}{2} = \left[-x^{-1} \ln x \right] + \int \frac{1}{x^2} \, dx = \frac{-\ln x}{x} - x^{-1} + C$$

$$t^2 = 2\left(\frac{1}{x}(-\ln x - 1) + C \right)$$

At $x=1$, $t=2$.

$$4 = 2\left(\frac{1}{1}(-\ln 1 - 1) + C \right) \Rightarrow 2 = -1 + C$$

$$\Rightarrow C = 3$$

So $t^2 = 2\left(-\frac{1}{x}(\ln x + 1) + 3 \right)$

7 (a) Express $\frac{4x+3}{(x-1)^2}$ in the form $\frac{A}{x-1} + \frac{B}{(x-1)^2}$

[3 marks]

$4x + 3 = A(x-1) + B = Ax + B - A$

Equating coefficients:

$A = 4$, $B - A = 3$

$\Rightarrow B - 4 = 3$ so $B = 7$

$$\frac{4x+3}{(x-1)^2} = \frac{4}{x-1} + \frac{7}{(x-1)^2}$$

7 (b) Show that

$$\int_3^4 \frac{4x+3}{(x-1)^2}\,dx = p + \ln q$$

where p and q are rational numbers.

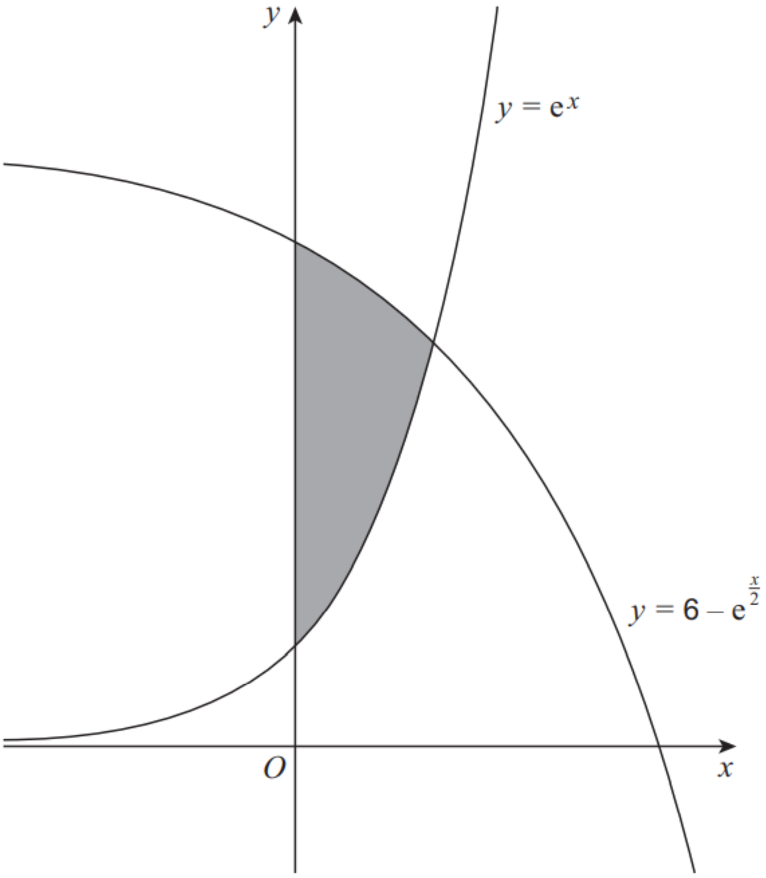
[5 marks]

$$\int_3^4 \frac{4x+3}{(x-1)^2}\,dx = \int_3^4 \frac{4}{x-1} + \frac{7}{(x-1)^2}\,dx$$

$$= \left[4\ln(x-1)\right]_3^4 + \left[-7(x-1)^{-1}\right]_3^4 = (4\ln 3 - 4\ln 2) + (-7(3)^{-1} + 7(2)^{-1})$$

$$= 4\ln\tfrac{3}{2} + \tfrac{7}{6}$$

15 The region enclosed between the curves $y = e^x$, $y = 6 - e^{\frac{x}{2}}$ and the line $x = 0$ is shown shaded in the diagram below.



Show that the exact area of the shaded region is

$$6 \ln 4 - 5$$

Fully justify your answer.

[10 marks]

$y = e^x$ and $y = 6 - e^{x/2}$

$e^x = 6 - e^{x/2}$

Let $Y = e^{x/2}$

$Y^2 = 6 - Y \Rightarrow Y^2 + Y - 6 = 0$

$(Y-2)(Y+3) = 0$

$Y = 2, \ Y = -3$

$e^{x/2} = 2, \ e^{x/2} = -3$

$x = \ln 4$ and $x \neq 2\ln -3$ as the domain of $\ln x$ is $x > 0$

$$\int_0^{\ln 2} 6 - e^{x/2} - e^x \, dx = \left[6x - 2e^{x/2} - e^x \right]_0^{\ln 4} = 6\ln 4 - 2e^{\frac{1}{2}\ln 4} - e^{\ln 4} + 2 + 1$$

$= 6 \ln 4 - 4 - 4 + 3 = 6 \ln 4 - 5$